STUDIES ON SEISMIC WAVES: II. RAYLEIGH WAVES IN A SUPERFICIAL LAYER*

C. Y. FU†

ABSTRACT

Lamb's method in the theory of the plate is extended to the case in which one of the surfaces is not free. The resulting determinantal relation is similar to that of Sezawa. It is then simplified and special cases of the frequency-velocity relation are discussed. Even when the thickness of the layer is as small as a wave length, the interaction of the upper and lower boundaries of the layer is quite slight and Rayleigh waves and Stoneley's waves may be discussed separately. A few points in connection with the application of this frequency relation to the ground roll problem are also discussed.

1. Introduction. In the theory of the simple Rayleigh waves, the medium is assumed to be homogeneous and semi-infinite in extent. The velocity of the surface waves so obtained depends on the material constants of the medium only and is independent of the frequency of the waves. On the other hand, if the medium is overlain by a layer of a different material, waves not only of the Rayleigh type but also of the Mock-Rayleigh type† will both exist in the layer. The disturbance resulting from the superposition of these two types of waves, when analyzed into harmonic components along the boundary, travels with velocities depending on the frequency. We have thus a dispersion of the Rayleigh waves, the latter being understood in the general sense. In seismic explorations, the existence of a superficial layer is by far more in accord with the experience and the circumstances for the generation of the simple Rayleigh waves are seldom realized. The dispersed waves make their appearance in groups whose velocities give the rates of the transmission of energy. Whether the wave groups recorded in the seismograms are entirely due to dispersion or not will not be discussed here, but the effect itself is inescapable if the theory of small displacements is maintained.

As an extension of Lord Rayleigh's original work on surface waves, the present problem was attempted a long time ago by Love‡ with the assumption of incompressibility. By the same method, the work was

* By permission of the United Geophysical Company, Pasadena, Calif.
† Part-time research Geophysicist in the United Geophysical Company.
followed up by Sezawa, Stoneley, Lee and some others, the compressibilities of the media being taken into consideration. However, the problem may also be approached from a different direction. The propagation of waves in an elastic plate bounded by free surfaces has been studied by Lord Rayleigh and also by Lamb. The method used by the latter was refined in his later paper with even more clarity and elegance. We have applied the method to the discussion of reflection and refraction of plane waves, but if we regard one of the two surfaces of the plate as being not free, we can easily extend Lamb's method to the present problem. Although it is actually a special case of our previous study, we propose to discuss the problem anew in more detail.

2. The Method of Normal Functions and the Frequency Equation.

Let the layer of thickness $H$ elastic constants $\lambda_1, \mu_1$ and density $\rho_1$ overlie a semi-infinite medium of constants $\lambda_2, \mu_2$ and $\rho_2$. Let the co-ordinate axes be chosen as shown in Fig. 1. Expressing the displacement components $u$ and $v$ by

$$u = \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y}$$

$$v = \frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x}$$

Fig. 1. Co-ordinate axes and notations.

---

we obtain the equations of motion in the steady state

\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} + h^2 \phi = 0
\]

\[
\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} + k^2 \psi = 0
\]  \hspace{1cm} (2)

where

\[
h^2(\lambda + 2\mu) = \mu k^2 = \rho \omega^2
\]  \hspace{1cm} (3)

and \(\omega\) is the angular frequency of the wave. Solutions which are periodic in \(x\) may be written in the forms \(\phi = \phi'e^{i\xi x}\) and \(\psi = \psi'e^{i\xi x}\). Then \(\phi'\) and \(\psi'\) satisfy

\[
\frac{d^2 \phi'}{dz^2} - \alpha^2 \phi' = 0
\]

\[
\frac{d^2 \psi'}{dz^2} - \beta^2 \psi' = 0
\]  \hspace{1cm} (4)*

respectively where

\[
\alpha^2 + h^2 = \beta^2 + k^2 = \xi^2.
\]  \hspace{1cm} (5)

The solutions of (4) are evidently of the forms

\[
\phi' = A \sinh \alpha z + B \cosh \alpha z
\]

\[
\psi' = C \sinh \beta z + D \cosh \beta z
\]  \hspace{1cm} (6)

It is to be understood that in all the solutions, the time factor \(e^{-i\omega t}\) has been omitted, and \(+\alpha\) and \(+\beta\) are positive. If we assume that there is no source of disturbance situated at \(z = -\infty\), the wave functions for the lower medium cannot contain terms of the forms \(e^{-\alpha z}\) and \(e^{-\beta z}\).

Hence for the lower medium, (6) degenerate into

\[
\phi' = E e^{\alpha z}, \quad \psi' = F e^{\beta z}.
\]  \hspace{1cm} (6a)

The stress components caused by the waves are

\[
T_{zz} = \mu(\xi^2 + \beta^2)\phi - 2i\mu\xi \frac{\partial \psi}{\partial z}
\]  \hspace{1cm} (7)

* Note the difference in signs with the equations in the previous study. This is due to (5).

** We write \(\sinh\) and \(\cosh\) for \(\sinh\) and \(\cosh\) respectively.
The vanishing of these at $z=H$ gives

$$
\begin{align*}
(\xi^2 + \beta_1^2)(A \sin \alpha_1 H + B \cos \alpha_1 H) - 2i\beta_1 \xi (C \sin \beta_1 H + D \cos \beta_1 H) &= 0 \\
2i\alpha_1 \xi (A \sin \alpha_1 H + B \cos \alpha_1 H) + (\xi^2 + \beta_1^2)(C \sin \beta_1 H + D \cos \beta_1 H) &= 0
\end{align*}
$$

where the subscript 1 refers to the quantities appropriate to the upper medium. The continuity of stress at $z=0$ leads to

$$
\begin{align*}
\mu_1(\xi^2 + \beta_1^2)B - 2i\mu_1 \beta_1 \xi C - \mu_2(\xi^2 + \beta_2^2)E + 2i\mu_2 \beta_2 \xi F &= 0 \\
2i\mu_1 \alpha_1 \xi A + \mu_1(\xi^2 + \beta_1^2)D - 2i\mu_2 \alpha_2 \xi E - \mu_2(\xi^2 + \beta_2^2)F &= 0
\end{align*}
$$

and that of displacement, to

$$
\begin{align*}
i\xi B + \beta_1 C - i\xi E - \beta_2 F &= 0 \\
i\xi A - i\xi D - \alpha_2 E + i\xi F &= 0
\end{align*}
$$

The six equations (8), (9), (10) are homogeneous in the six constants $A, B, C, D, E, F$. In order that they be soluble for five independent ratios among these constants, it is necessary and sufficient that the determinant of these equations should vanish. So we must have

$$
\begin{vmatrix}
(\xi^2 + \beta_1^2) \sin \alpha_1 H & -2i\beta_1 \xi \sin \beta_1 H & (\xi^2 + \beta_1^2) \cos \alpha_1 H & -2i\beta_1 \xi \cos \beta_1 H & 0 & 0 \\
2i\alpha_1 \xi \sin \alpha_1 H & (\xi^2 + \beta_1^2) \cos \alpha_1 H & 2i\alpha_1 \xi \cos \alpha_1 H & (\xi^2 + \beta_1^2) \sin \alpha_1 H & 0 & 0 \\
\mu_1(\xi^2 + \beta_1^2) & -2i\mu_1 \beta_1 \xi & 0 & 0 & -\mu_1(\xi^2 + \beta_1^2) & 2i\mu_1 \beta_1 \xi \\
i\xi & \beta_1 & 0 & 0 & -i\xi & -\beta_1 \\
o & o & zim\alpha_1 \xi & \mu_1(\xi^2 + \beta_1^2) & -zim\alpha_1 \xi & -\mu_1(\xi^2 + \beta_1^2) \\
o & o & \alpha_1 & -i\xi & -\alpha_2 & i\xi
\end{vmatrix} = 0. \quad (11)
$$

This is a relation between $\xi$ and $\omega$. If we denote the apparent velocity of the wave along the surface by $V_a$ and the apparent wave length by $\lambda$, then

$$
\xi = \omega / V_a = 2\pi / \lambda.
$$

Hence (11) may also be regarded as a relation between $V_a$ and $\lambda$. The graph of this is the dispersion curve.

To facilitate discussions, let us make the following abbreviations which are slightly different from those of Love and Lee:

$$
\begin{align*}
X &= 2\mu_2 \xi^2 - \mu_1(\xi^2 + \beta_1^2), & Z &= \mu_1(\xi^2 + \beta_1^2) - \mu_2(\xi^2 + \beta_2^2), \\
Y &= 2\mu_1 \xi^2 - \mu_2(\xi^2 + \beta_2^2), & W &= 2\xi^2(\mu_2 - \mu_1).
\end{align*}
$$

Then (11) may be reduced to

$$
\xi \eta' - \xi' \eta = 0 \quad (13)
$$
where

\[ \xi = (\xi^2 + \beta_1^2) \left[ \left( \frac{\alpha_2}{\alpha_1} \right) X \sh \alpha_1 H + Y \ch \alpha_1 H \right] - 2 \left[ \alpha_2 \beta_1 W \sh \beta_1 H + \xi^2 Z \ch \beta_1 H \right] \]

\[ \eta' = (\xi^2 + \beta_1^2) \left[ \left( \frac{\beta_2}{\beta_1} \right) X \sh \beta_1 H + Y \ch \beta_1 H \right] - 2 \left[ \alpha_1 \beta_2 W \sh \alpha_1 H + \xi^2 Z \ch \alpha_1 H \right] \]

\[ \xi' = (\xi^2 + \beta_1^2) \left[ \left( \frac{\beta_2}{\xi} \right) W \ch \alpha_1 H + \left( \frac{\xi}{\alpha_1} \right) Z \sh \alpha_1 H \right] - 2 \left[ \beta_2 \xi X \ch \beta_1 H + \beta_1 \xi Y \sh \beta_1 H \right] \]

\[ \eta = (\xi^2 + \beta_1^2) \left[ \left( \frac{\alpha_2}{\xi} \right) W \ch \beta_1 H + \left( \frac{\xi}{\beta_1} \right) Z \sh \beta_1 H \right] - 2 \left[ \alpha_2 \xi X \ch \alpha_1 H + \alpha_1 \xi Y \sh \alpha_1 H \right]. \]

(14)

The calculation of the dispersion curve from (13) is quite laborious, but the correctness of the equation can be checked by considering its special cases. We notice that when \( H \) approaches zero, the formula should reduce to the Rayleigh wave equation for the lower medium. On the other hand, when \( H \) is large compared with the wave length, the interaction of the upper boundary and the interface would be slight and we should expect to obtain both the Rayleigh wave equation for the layer and the Stoneley-wave equation at the interface from (13). This is indeed the case as will be shown presently.

3. The Simple Rayleigh Waves and the Stoneley Waves. (i) Simple Rayleigh waves in the medium. When \( H \) approaches zero,

\[ \sh \alpha_1 H, \sh \beta_1 H \to 0; \ch \alpha_1 H, \ch \beta_1 H \to 1, \]

and expressions (14) reduce to

\[ \xi \sim (\xi^2 + \beta_1^2) Y - 2 \xi^2 Z \]

\[ \eta' \sim (\xi^2 + \beta_1^2) Y - 2 \xi^2 Z \]

\[ \xi' \sim (\xi^2 + \beta_1^2) (\beta_2/\xi) W - 2 \beta_2 \xi X \]

\[ \eta \sim (\xi^2 + \beta_1^2) (\alpha_2/\xi) W - 2 \alpha_2 \xi X. \]

(15)

Hence (13) becomes

\[ [(\xi^2 + \beta_1^2) Y - 2 \xi^2 Z]^2 - (\alpha_2 \beta_2/\xi^2) [(\xi^2 + \beta_1^2) W - 2 \xi^2 X] = 0. \]

Substituting \( X, Y, Z, W \) from (12), we obtain

\[ (\xi^2 + \beta_2^2)^2 [\mu_2 (\xi^2 - \beta_1^2)]^2 - 4 \alpha_2 \beta_2 \xi^2 [\mu_2 (\xi^2 - \beta_1^2)]^2 = 0. \]
Since \( h \neq 0, k \neq 0 \), hence \( \xi \neq \beta_1 \) and we have
\[
(\xi^2 + \beta_1^2)^2 - 4 \alpha_1 \beta_2 \xi^2 = 0
\]
or
\[
(2 \xi^2 - k_1^2)^2 - (\xi^2 - h_1^2)^{1/2}(\xi^2 - k_1^2)^{1/2} \xi^2 = 0 \tag{16}
\]
which is precisely the simple Rayleigh wave equation for the lower medium.

(ii) Simple Rayleigh waves in the layer. When \( H \) is large compared with the apparent wave length \( \lambda \),
\[
\text{sh } \alpha_1 H \sim \text{ch } \alpha_1 H \sim \frac{1}{2} e^{\alpha_1 H}; \quad \text{sh } \beta_1 H \sim \text{ch } \beta_1 H \sim \frac{1}{2} e^{\beta_1 H},
\]
and (14) become
\[
\xi' \sim \frac{1}{2} (\xi^2 + \beta_1^2) ((\alpha_2 / \alpha_1) X + Y) e^{\alpha_1 H} - (\alpha_2 \beta_1 W + \xi Z^2) e^{\beta_1 H}
\]
\[
\eta' \sim \frac{1}{2} (\xi^2 + \beta_1^2) ((\beta_2 / \beta_1) X + Y) e^{\beta_1 H} - (\alpha_1 \beta_2 W + \xi Z^2) e^{\alpha_1 H}
\]
\[
\xi'' \sim \frac{1}{2} (\xi^2 + \beta_1^2) ((\xi / \alpha_1) X + (\xi / \beta_1) Z) e^{\alpha_1 H} - (\beta_2 \xi X + \beta_1 \xi Y) e^{\beta_1 H}
\]
\[
\eta'' \sim \frac{1}{2} (\xi^2 + \beta_1^2) ((\alpha_2 / \xi) X + (\xi / \beta_1) Z) e^{\beta_1 H} - (\alpha_2 \xi X + \alpha_1 \xi Y) e^{\alpha_1 H}
\]
Substituting in (13) and simplifying, we obtain
\[
[(\xi^2 + \beta_1^2)^2 - 4 \alpha_1 \beta_1 \xi^2] \left[ (\alpha_2 X + \alpha_1 Y) (\beta_2 X + \beta_1 Y) \xi^2 - (\alpha_2 \beta_1 W + \xi Z^2) (\alpha_1 \beta_2 W + \xi Z^2) \right] = 0 \tag{18}
\]
The first factor equated to zero gives
\[
(2 \xi^2 - k_1^2)^2 - (\xi^2 - h_1^2)^{1/2}(\xi^2 - k_1^2)^{1/2} \xi^2 = 0 \tag{19}
\]
which is the simple Rayleigh wave equation for the layer.

(iii) The Stoneley-waves. The second factor in (18) contains the material constants of both media but is independent of \( H \). This strongly suggests a disturbance along the interface. It was actually first studied by Love\(^8\) but is also known as the Stoneley wave.\(^9\) To confirm this conjecture, and to facilitate comparison, we shall derive the Stoneley-wave equation by the present method, which is quite different from Stoneley's derivation.

\(^8\) Love, Geodynamics, §190.
In Fig. 2, are shown two semi-infinite media in contact at a plane \( z = 0 \). The problem is to find the frequency relation for the waves traveling along the interface and dying out in both directions away from the interface. Using the same notations as before, we proceed to solve the equations (4) under the condition that both media are semi-infinite. Then instead of (6), the solutions would be of the forms

\[
\phi_1 = Ae^{-az}, \quad \psi_1 = Ce^{-b_1z},
\]

\[
\phi_2 = Ee^{-az}, \quad \psi_2 = Fe^{b_2z}.
\]

The continuities of the displacement and the stress at \( z = 0 \) lead to

\[
\mu_1(\xi^2 + \beta_1^2)A + 2i\mu_1\beta_1\xi C - \mu_2(\xi^2 + \beta_2^2)E + 2i\mu_2\beta_2\xi F = 0
\]

\[
-2i\mu_1\alpha_1\xi A + \mu_1(\xi^2 + \beta_1^2)C - 2i\mu_2\alpha_2\xi E - \mu_2(\xi^2 + \beta_2^2)F = 0
\]

\[
i\xi A - \beta_1 C - i\xi E - \beta_2 F = 0
\]

\[-\alpha_1 A - i\xi C - \alpha_2 E + i\xi F = 0.
\]

Equating the determinant of these equations to zero as before, we have

\[
\begin{vmatrix}
\mu_1(\xi^2 + \beta_1^2) & 2i\mu_1\beta_1\xi & -\mu_2(\xi^2 + \beta_2^2) & 2i\mu_2\beta_2\xi \\
-2i\mu_1\alpha_1\xi & \mu_1(\xi^2 + \beta_1^2) & -2i\mu_2\alpha_2\xi & -\mu_2(\xi^2 + \beta_2^2) \\
i\xi & -\beta_1 & -i\xi & -\beta_2 \\
-\alpha_1 & -i\xi & -\alpha_2 & i\xi
\end{vmatrix} = 0. \tag{21}
\]

With the abbreviations (12), it can be shown that (21) may be reduced to

\[
(\alpha_2\xi X + \alpha_1\xi Y)(\beta_2\xi X + \beta_1\xi Y) - (\alpha_1\beta_2 W + \xi Z)(\alpha_2\beta_1 W + \xi Z) = 0. \tag{22}
\]
a factor $1 - \alpha_1\beta_1/\xi^2$ being cancelled since $\xi^2 \neq \alpha_1\beta_1$. The left hand side of this equation is the same as the second factor in (18).

4. The First Perturbation Due to a Thin Layer. We have examined in the previous section two extreme cases: that of a vanishing layer or infinitely long waves and that of a very thick layer or very short waves. However, the effect of a thin, but finite layer on the behavior of the Rayleigh waves is more important. We shall calculate this effect by regarding it as a perturbation on the Rayleigh waves in the lower medium.

When $H$ is small,

$$\sin \alpha_1 H \sim \alpha_1 H, \quad \sin \beta_1 H \sim \beta_1 H, \quad \cos \alpha_1 H \sim \cos \beta_1 H \sim 1$$

$$\xi = (\xi^2 + \beta_2^2)Y - 2\xi^2Z + [(\xi^2 + \beta_1^2)X - 2\beta_1^2W]\alpha_2 H$$

$$\eta' = (\xi^2 + \beta_2^2)Y - 2\xi^2Z + [(\xi^2 + \beta_1^2)X - 2\alpha_1^2W]\beta_2 H$$

$$\xi' = (\xi^2 + \beta_2^2)Y(\beta_2/\xi)W - 2\beta_2\xi X + [(\xi^2 + \beta_1^2)Z - 2\beta_1^2Y]\xi H$$

$$\eta = (\xi^2 + \beta_2^2)Y(\alpha_2/\xi)W - 2\alpha_2\xi X + [(\xi^2 + \beta_1^2)Z - 2\alpha_1^2Y]\xi H.$$  

Neglecting all terms of higher orders than the first power in $H/\lambda$, we may write (13) after simplification as

$$\xi\eta' - \xi'\eta = [(\xi^2 + \beta_2^2)^2 - 4\alpha_2\beta_2\xi^2][\mu_2(\xi^2 - \beta_1^2)]^2$$

$$+ \mu_1\mu_2(\xi^2 - \beta_1^2)(\xi^2 - \beta_2^2)(\alpha_2 + \beta_2)H$$

$$+ 4\mu_1\mu_2\xi^2(\xi^2 - \beta_1^2)(\xi^2 - \beta_2^2)(\beta_1^2 - \alpha_1^2)\beta_2 H = 0.$$  

Since $\xi \neq \beta_1$, this may be reduced to

$$(\xi^2 + \beta_2^2)^2 - 4\alpha_2\beta_2\xi^2 + \frac{\mu_1}{\mu_2}(\xi^2 - \beta_1^2)(\xi^2 - \beta_2^2)(\alpha_2 + \beta_2)H$$

$$+ 4\mu_1\xi^2(\xi^2 - \beta_1^2)(\beta_1^2 - \alpha_1^2)\beta_2 H = 0. \quad (23)$$

Recall (5) and let $\gamma_1$ and $\gamma_2$ be the ratios of the velocities of the dilatational to those of the shear waves in the two media, i.e.,

$$V_1^2 = \gamma_1^2v_1^2; \quad V_2^2 = \gamma_2^2v_2^2. \quad (24)$$

Then $\xi^2 - \beta_1^2 = k_1^2$, $\xi^2 - \beta_2^2 = k_2^2$, $(\alpha_1^2 - \beta_1^2)/(\xi^2 - \beta_1^2) = (\gamma_1^2 - 1)/\gamma_1^2$ and (23) is reduced to

$$[(\xi^2 + \beta_2^2)^2 - 4\alpha_2\beta_2\xi^2]$$

$$+ H \frac{\mu_1 k_1^2}{\mu_2} \left[ k_1^2(\alpha_2 + \beta_2) - 4(\gamma_1^2 - 1)\xi^2 \right] \beta_2 = 0. \quad (23a)$$
which is linear in $H$. The first term equated to zero is simply the Rayleigh-wave equation for the lower medium. Let its root be $\xi_0$. The effect of the second term in (23a) on its root is $\xi$ is small because $H$ is supposed to be small. Write $\xi = \xi_0 + \delta \xi$. Expanding the first term in the neighborhood of $\xi_0$ by Taylor's theorem, we have

$$f(\xi) = (\xi^2 + \beta_2 \xi) = f(\xi_0) + (\partial f/\partial \xi)\delta \xi + \cdots$$

$$\approx 4\xi_0\delta \xi \left[ 2(2\xi_0^2 - k_2^2 - \alpha_2 \beta_2) - \beta_2 \xi_0^2/\alpha_2 - \alpha_2 \xi_0^2/\beta_2 \right] + \cdots$$

$$\approx 4\xi_0^3 \delta \xi \left[ 4 - 2k_2^2/\xi_0^2 - 2\alpha_2 \beta_2/\xi_0^2 - \beta_2/\alpha_2 - \alpha_2/\beta_2 \right]$$

where $\alpha_2, \beta_2$ are to be evaluated for $\xi = \xi_0$. This root of the simple Rayleigh wave equation is generally solved in terms of $k_2$ which is known when the frequency of the wave and the nature of the medium are given. Let it be $m$ so that

$$\xi_0^2 = m^2 k_2^2$$

and therefore

$$\frac{\alpha_2^2}{\xi^2} = \frac{(m^2 - 1/\gamma_2^2)}{m^2}, \quad \frac{\beta_2^2}{\xi^2} = \frac{(m^2 - 1/\gamma_2^2)}{m^2},$$

$$\frac{\alpha_2^2}{\beta_2^2} = \frac{(m^2 - 1/\gamma_2^2)}{(m^2 - 1)}. \quad (26)$$

Then

$$\delta \xi \left[ 4 - \frac{2}{m^2} \left\{ (m^2 - 1)(m^2 - 1/\gamma_2^2) \right\}^{1/2} - \left\{ \frac{m^2 - 1}{m^2 - 1/\gamma_2^2} \right\}^{1/2} \right] = \frac{H}{4} \left[ \frac{k_2^2}{m^2} \right] - \frac{2m^2 - 1 - 1/\gamma_2^2}{m^2}$$

$$- \frac{k_2^2(\gamma_2^2 - 1)}{\gamma_2^2} \left\{ \frac{m^2 - 1}{m^2} \right\}^{1/2} \right] \quad (27)$$

The expression in the bracket is known when the media are given and the simple Rayleigh wave equation for the lower medium solved.

To recapitulate, we have

(A) $$(2\xi_0^2 - k_2^2)^2 - 4(\xi_0^2 - k_2^2)^{1/2}(\xi_0^2 - k_2^2)^{1/2}\xi_0^2 = 0 \quad \text{(Rayleigh Wave Equation)}$$

(B) $$4 - \frac{2}{m^2} (1 + ab) - \frac{a}{b} - \frac{b}{a} \right] \delta \xi$$

$$= \frac{H}{4} \left[ \frac{k_2^2(\gamma_1^2 - 1)}{\gamma_1^2} \right] - \frac{k_2^2(a^2 + b^2)}{m^2} \quad \text{(First Perturbation)}$$

(C) $\xi = \xi_0 + \delta \xi$
where

\[ a^2 = m^2 - 1, \quad b^2 = m^2 - 1/\gamma z^2. \]  \hspace{1cm} (28)

From (B) it is seen that \( \delta \xi \) is proportional to \( \omega^2 \) for a given \( V_o \), while \( \xi \) is proportional to \( \omega \). Hence (C), as a function of \( V_o \) and \( \lambda \), is of the form

\[ c_1 V_o^2 + \lambda = c_2 V_o \]

which is a hyperbolic arc, \( c_1, c_2 \) being constants. It should be emphasized here that the present approximation is good only when \( H/\lambda \ll 1 \).

5. Discussions. The natures of the waves depend primarily on the four parameters \( \alpha_1, \beta_1, \alpha_2, \beta_2 \). Our analysis so far is based on the assumption that all of them are real. Introducing the identities \( \sin ix = i \sin x \) and \( \cos ix = \cos x \) into (14), we see that the relation (13) is still real when \( \beta_1 \) or both \( \alpha_1, \beta_1 \) become imaginary. It is easy to see that

\[ \alpha_1 H = (2\pi H/\lambda)(1 - V_o^2/V_1^2)^{1/2}, \]

\[ \beta_1 H = (2\pi H/\lambda)(1 - V_o^2/v_1^2)^{1/2}, \]

\[ \alpha_2 H = (2\pi H/\lambda)(1 - V_o^2/V_2^2)^{1/2}, \]

\[ \beta_2 H = (2\pi H/\lambda)(1 - V_o^2/v_2^2)^{1/2}. \]  \hspace{1cm} (29)

For these to be real or imaginary, the criterion is determined by the value of \( V_o \) relative to \( V_1, V_2, v_1, v_2 \), which are in turn determined by the material constants of the media. The case when all these parameters are imaginary has been examined in a previous study. We are concerned here only with the Rayleigh waves and their transition to other types.

(i) Overlying low-speed layer. This is to approximate the weathering layer (idealized, of course). When \( \lambda \) (or alternatively \( H \)) is varied from 0 to \( \infty \), \( V_o \) varies between the values of the simple Rayleigh wave velocities appropriate to the two media and which we will designate by \( V(R_1) \) and \( V(R_2) \). It has been shown both by Love and by Sezawa\(^{10}\) that \( V_o \) increases with \( \lambda \). Without making detailed calculations, we can sketch the dispersion curve as in Fig. 3. The part of the curve for large \( \lambda \) is asymptotic to the line \( V_o = V(R_2) \). This is confirmed by the fact that this part of the curve is hyperbolic. When \( \lambda \) is very small, the curve starts from \( V_o = V(R_1) \) according to (19). From Sezawa’s results,

\(^{10}\) Op. cit.
the curvature of this part of the curve is concave upward. This fact is of some significance as will be shown later.

The curve is drawn with the assumption that the simple Rayleigh-wave velocity in the lower medium is greater than the velocity of the distortional waves in the layer, \( V(R_2) > v_1 \). This is plausible because \( V(R_2) \) is only slightly smaller than \( v_2 \). This being the case, the expression of \( \beta H \) in (29) shows that when \( V_a > v_1 \), \( \beta_1 \) becomes imaginary. From (6), this indicates that the distortional waves will then exist only as body waves. For apparent velocities greater than \( v_1 \), the Rayleigh type of waves must be dilatational in nature. When \( V_a \) is greater than \( V_1 \), both \( \alpha_1 \) and \( \beta_1 \) become imaginary, and we have only body waves. There is actually a second branch of the dispersion curves which holds for body waves only, but we need not be concerned with it here.

From (29), when \( V_a \) is increased, both \( \alpha_1 \) and \( \beta_1 \) decrease (\( \lambda \) increases). This would mean that faster or longer waves penetrate deeper into the layer. Here, caution should be taken in the use of the notion of frequency \( \nu \) defined by \( \nu \lambda = V_a \). In Fig. 3, the dispersion curve shows a point of inflection at \( \lambda = \lambda_i \). For wave lengths longer than \( \lambda_i \), \( V_a \) increases less than a mere linear increase and longer wave means lower frequency. When \( \lambda < \lambda_i \), \( V_a \) increases faster than a linear increase and longer waves means higher frequency.

A few words concerning the order of magnitudes seem to be in
place. Our approximations (16b) are based on the assumptions that
$e^{\alpha_1 H} \gg e^{-\alpha_1 H}$ and $e^{\beta_1 H} \gg e^{-\beta_1 H}$. It is sufficient to discuss only the quantity

$$\beta_1 H = \left( \frac{2\pi H}{\lambda} \right) \left\{ 1 - \left( \frac{V_a}{V_1} \right)^2 \right\}^{1/2}.$$ 

Since there are two variables $\lambda/H$ and $V_a/v_1$, in this expression, accurate estimate can be made only by solving (13). However, we may get a general idea by resorting to Sezawa's work. There, it was found that when $H > \lambda$, $V_a$ is very near to $V(R_1)$. For all the media that may possibly exist, the minimum value of $\varepsilon^2/V^2(R)$ is $1.0957$. Let us take this value for $v_1^2/V_a^2$. Then $\beta_1 H = 1.857(H/\lambda)$ and we have the following table:

<table>
<thead>
<tr>
<th>Table 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H/\lambda$</td>
</tr>
<tr>
<td>$\beta_1 H$</td>
</tr>
<tr>
<td>$\beta_1 H$</td>
</tr>
<tr>
<td>$\beta_1 H$</td>
</tr>
</tbody>
</table>

Even for $H = \lambda$, the approximation is not bad and for $H = 3\lambda$ the approximation is really excellent. Hence we may conclude that for a low speed layer of thickness larger than the wave length in which we are interested, the effect of the lower medium on the surface waves is insignificant.

(ii) Overlying high-speed layer. A well known approximation of this case is caliche overlying clay. Here the approximations (15) and (17) still hold and the dispersion curve would be of the form in Fig. 4. It starts from $V_a = V(R_1)$ when $\lambda$ is very small and approaches $V(R_2)$ when $\lambda$ is very large. But a point of interest is the fact that the curve would intersect the line $V_a = v_2$ at $\lambda = \lambda_c$ when $\lambda < \lambda_c$, $V_a > v_2$. From (29), $\beta_2$ will be imaginary. The physical significance of this is that no distortional waves of the Rayleigh type with wave lengths less than $\lambda_c$ can exist below the lower boundary of the layer. If, further, the media are such that $V(R_1) > V_2$, there will exist a length $L$ such that waves shorter than it can be propagated in the lower medium only as body waves. The depth of penetration is increased when the wave length is increased. But this refers to the surface waves in which the main part of the energy resides. They are usually annoying and it is the body waves which are more useful. The use of a high speed dynamite will probably help increase the efficiency of exploration through a high speed layer.

(iii) Remarks on ground rolls and group velocity. We do not propose to exploit here the various causes of ground rolls, but a few words concerning observation seem to be worth mentioning. On account of dispersion, what we have observed in the records as surface waves gives only the group velocity and should not be compared directly with Fig. 3. To get the group velocity analytically from (13) is quite involved. However, we may make use of a graphical method due to Lamb\(^\text{12}\) when the \(V_a - \lambda\) curve has been calculated. At any point \(P\) on the curve in Fig. 5, let us draw the tangent. The intercept on the \(V\)-axis gives the group velocity \(U\) corresponding to \(\lambda_a\). This is obvious from the definition

\[
U = V - \lambda (dV/d\lambda) \quad (30)
\]

According to this construction, it is seen that when \(V_a\) increases with \(\lambda\), the velocity observed from the records may not do so. It depends on the curvature of the \(V_a - \lambda\) curve. When it is as shown in Fig. 5, the group velocity first decreases as \(\lambda\) is increased. It reaches a minimum at the point of inflection \(I(\lambda_i, V_i)\) and then increases with the wave length. Since this minimum group velocity is stationary with respect to the wave length, the waves corresponding to this velocity should be dominant in the record and preserve their forms.

It has often been taken for granted that the surface waves have a

constant velocity. This would mean \( U = \text{constant} \). From (30), it is obvious that \( V \) must then be a linear function of \( \lambda \) i.e.,

\[
V = a + b\lambda.
\]  

From the dispersion curve, this is approximately satisfied when \( \lambda \) is in the neighborhood of \( \lambda \), where \( U \) is a minimum or when \( \lambda \) is very large where \( U \) and \( V \) are nearly equal.

Fig. 5. Graphical calculation of group velocity.